Stochastic maximum principle for optimal control of SPDEs. *

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Abstract.

In this note, we give the stochastic maximum principle for optimal control of stochastic PDEs in the general case (when the control domain need not be convex and the diffusion coefficient can contain a control variable).

1 Introduction

The problem of finding necessary optimality conditions for stochastic optimal control problems (generalizing in this way the Pontryagin maximum principle to the stochastic case) has been solved in great generality, in the classical finite dimensional case, in the well known paper by S. Peng [4]. The author allows the set of control actions to be non-convex and the diffusion coefficient to depend on the control; consequently he is led to introducing the equations for the second variation process and for its dual. As far as infinite dimensional equations are concerned the cases in which the control domain is convex or diffusion does not depend on the control have been treated in [1, 2]. On the contrary in the general case (when the control domain need not be convex and the diffusion coefficient can contain a control variable) existing results are limited to abstract evolution equations under assumptions that are not satisfied by the large majority of concrete stochastic PDEs (for instance the case of Nemitsky operators on L^p spaces is not covered, see [5, 3]). Here we formulate a controlled parabolic stochastic PDE in a semi-abstract way and show how the specific regularity properties on the semigroup corresponding to the differential operator can be used to treat the case of Nemitsky-type coefficients. The key point is the explicit characterization of the second dual process in term of a suitable quadratic functional, see Definition 4.1.

*Supported by Marie Curie ITN Call: FP7-PEOPLE-2007-1-1-ITN, no. 213841-2: *Deterministic and Stochastic Controlled Systems and Applications*

2 Formulation of the optimal control problem

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded open set with regular boundary. We consider the following controlled SPDE formulated in a partially abstract way in the state space $H = L^2(\mathcal{O})$ (norm $|\cdot|$, scalar product $\langle \cdot, \cdot \rangle$):

$$\begin{cases} dX_t(x) &= AX_t(x)\,dt + b(x, X_t(x), u_t)\,dt + \sum_{j=1}^{m} \sigma_j(x, X_t(x), u_t)\,d\beta_t^j, \qquad t \in [0,T], x \in \mathcal{O}, \\ X_0(x) &= x_0(x), \end{cases} \tag{2.1}$$

and the following cost functional:

$$J(u) = \mathbb{E} \int_0^T \int_{\mathcal{O}} l(x, X_t(x), u_t)\,dx\,dt + \mathbb{E} \int_{\mathcal{O}} h(x, X_T(x))\,dx.$$

We work in the following setting.

Hypothesis 2.1 *1. A is the realization of a partial differential operator with appropriate boundary conditions. We assume that A is the infinitesimal generator of a strongly continuous semigroup e^{tA}, $t \geq 0$, in H. Moreover, for every $p \in [2, \infty)$ and $t \in [0,T]$, $e^{tA}(L^p(\mathcal{O})) \subset L^p(\mathcal{O})$ with $\|e^{tA} f\|_{L^p(\mathcal{O})} \leq C_{p,T} \|f\|_{L^p(\mathcal{O})}$ for some constants $C_{p,T}$ independent of t and f. Finally the restriction of e^{tA}, $t \geq 0$, to $L^4(\mathcal{O})$ is an analytic semigroup with domain of the infinitesimal generator compactly embedded in $L^4(\mathcal{O})$.*

2. *$(\beta_t^1, \ldots, \beta_t^m)$, $t \geq 0$ is a standard m-dimensional Wiener process on a complete probability space $(\Omega, \mathcal{E}, \mathbb{P})$ and we denote by $(\mathcal{F}_t)_{t \geq 0}$ its natural (completed) filtration. All stochastic processes will be progressively measurable with respect to $(\mathcal{F}_t)_{t \geq 0}$.*

3. *$b, \sigma_j (j = 1, ..., m), l : \mathcal{O} \times \mathbb{R} \times U \to \mathbb{R}$ and $h : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ are measurable functions. We assume that they are continuous with respect to the third variable (the control variable), of class C^2 with respect to the second (the state variable), and bounded together with their first and second derivative with respect to the second variable.*

4. *the set of admissible control actions is a separable metric space U and an admissible control u is a (progressive) process with values in U.*

Under the above conditions, for every control u there exists a unique mild solution, i.e. a continuous process in H such that, $\mathbb{P}$-a.s.

$$X_t = e^{tA} x_0 + \int_0^t e^{(t-s)A} b(\cdot, X_s(\cdot), u_s)\,ds + \int_0^t e^{(t-s)A} \sigma_j(\cdot, X_s(\cdot), u_s)\,d\beta_s^j, \quad t \in [0,T].$$

3 Expansions of the solution and of the cost

We assume that an optimal control $\bar{u}$ exists and denote by $\bar{X}$ the corresponding optimal state. We introduce the spike variation: we fix an arbitrary interval $[\bar{t}, \bar{t}+\epsilon] \subset (0,T)$ and an arbitrary U-valued, $\mathcal{F}_{\bar{t}}$-measurable random variable v define the following perturbation of $\bar{u}$: $u_t^\epsilon = v I_{[\bar{t}, \bar{t}+\epsilon]}(t) + \bar{u}_t I_{[\bar{t}, \bar{t}+\epsilon]^c}(t)$ and denote by X^ϵ the solution of the state equation (2.1) with control $u = u^\epsilon$.

We introduce two linear equations corresponding to first and second expansion of X^ϵ with respect to ϵ (both equations are understood in the mild sense). In the following, derivatives with respect to the state variable will be denoted $b', b'', \sigma', \sigma''$ and

$$\delta b_t(x) = b(x, \bar{X}_t(x), u_t^\epsilon) - b(x, \bar{X}_t(x), \bar{u}_t), \qquad \delta\sigma_{jt}(x) = \sigma_j(x, \bar{X}_t(x), u_t^\epsilon) - \sigma_j(x, \bar{X}_t(x), \bar{u}_t),$$

$$\delta b'_t(x) = b'(x, \bar{X}_t(x), u_t^\epsilon) - b'(x, \bar{X}_t(x), \bar{u}_t), \qquad \delta\sigma'_{jt}(x) = \sigma'_j(x, \bar{X}_t(x), u_t^\epsilon) - \sigma'_j(x, \bar{X}_t(x), \bar{u}_t).$$

Consider

$$\begin{cases} dY_t^\epsilon(x) &= \Big[AY_t^\epsilon(x) + b'(x, \bar{X}_t(x), \bar{u}_t) \cdot Y_t^\epsilon(x) \Big] dt + \sigma'_j(x, \bar{X}_t(x), \bar{u}_t) \cdot Y_t^\epsilon(x)\,d\beta_t^j + \delta b_t(x)\,dt + \delta\sigma_{jt}(x)\,d\beta_t^j \\ Y_0^\epsilon(x) &= 0 \end{cases} \tag{3.1}$$

$$
\begin{cases}
dZ_t^\epsilon(x) = \Big[AZ_t^\epsilon(x) + b'(x,\bar X_t(x),\bar u_t)\cdot Z_t^\epsilon(x)\Big]\,dt + \sigma_j'(x,\bar X_t(x),\bar u_t)\cdot Z_t^\epsilon(x)\,d\beta_t^j \\
\qquad +\Big[\tfrac12 b''(x,\bar X_t(x),\bar u_t)\cdot Y_t^\epsilon(x)^2 + \delta b_t'(x)\cdot Y_t^\epsilon(x)\Big]dt + \Big[\tfrac12 \sigma_j''(x,\bar X_t(x),\bar u_t)\cdot Y_t^\epsilon(x)^2 + \delta\sigma_{jt}'(x)\cdot Y_t^\epsilon(x)\Big]d\beta_t^j \\
Z_0^\epsilon(x) = 0
\end{cases}
\tag{3.2}
$$

We notice that to formulate the second equation in H we need to show that the first admits solutions in $L^4(\mathcal{O})$.

The following proposition states existence and uniqueness of the solution to the above equation in all spaces $L^p(\mathcal{O})$ together with the estimate of their dependence with respect to ϵ. The proof is technical but based on standard estimates and we omit it.

Proposition 3.1 *Equations (3.1) and (3.2) admit a unique continuous mild solution. Moreover for all $p \geq 2$*

$$
\sup_{t\in[0,T]} \left(\sqrt{\epsilon}^{-1}(\mathbb{E}\|Y_t^\epsilon\|^p_{L^p(\mathcal{O})})^{1/p} + \epsilon^{-1}(\mathbb{E}\|Z_t^\epsilon\|^p_{L^p(\mathcal{O})})^{1/p}\right) \leq C_p, \qquad \sup_{t\in[0,T]} \left(\mathbb{E}\|X_t^\epsilon - \bar X_t - Y_t^\epsilon - Z_t^\epsilon\|_H^2\right)^{1/2} = o(\epsilon).
$$

As far as the cost is concerned we set $\delta l_t(x) = l(x,\bar X_t(x),u_t^\epsilon) - l(x,\bar X_t(x),\bar u_t)$ and prove that

Proposition 3.2

$$
J(u^\epsilon) - J(\bar u) = \mathbb{E}\int_0^T\int_{\mathcal{O}} \delta l_t(x)\,dx\,dt + \Delta_1^\epsilon + \Delta_2^\epsilon + o(\epsilon),
$$

where

$$
\begin{aligned}
\Delta_1^\epsilon &= \mathbb{E}\int_0^T\int_{\mathcal{O}} l'(x,\bar X_t(x),\bar u_t)(Y_t^\epsilon(x)+Z_t^\epsilon(x))\,dx\,dt + \mathbb{E}\int_{\mathcal{O}} h'(x,\bar X_T(x))(Y_T^\epsilon(x)+Z_T^\epsilon(x))\,dx, \\
\Delta_2^\epsilon &= \frac12\mathbb{E}\int_0^T\int_{\mathcal{O}} l''(x,\bar X_t(x),\bar u_t)\,Y_t^\epsilon(x)^2\,dx\,dt + \frac12\mathbb{E}\int_{\mathcal{O}} h''(x,\bar X_T(x))\,Y_T^\epsilon(x)^2\,dx.
\end{aligned}
$$

4 The first and second adjoint processes

The following proposition is special case of a result in [2]:

Proposition 4.1 *Let A^* be the $L^2(\mathcal{O})$-adjoint operator of A. There exists a unique $m+1$-tuple of $L^2(\mathcal{O})$ processes (p,q_j), with p continuous and $\mathbb{E}\sup_{t\in[0,T]}|p_t|^2 + \mathbb{E}\int_0^T |q_{jt}|^2\,dt < \infty$, that verify (in a mild sense) the backward stochastic differential equation:*

$$
\begin{cases}
-dp_t(x) = -q_{jt}(x)\,d\beta_t^j + \Big[A^*p_t(x) + b'(x,\bar X_t(x),\bar u_t)\cdot p_t(x) + \sigma_j'(x,\bar X_t(x),\bar u_t)\cdot q_{jt}(x) + l'(x,\bar X_t(x),\bar u_t)\Big]\,dt \\
p_T(x) = h'(x,\bar X_T(x)).
\end{cases}
$$

The following proposition formally follows from Proposition 3.2 computing the Itô differentials $d\int_{\mathcal{O}} Y_t^\epsilon(x)p_t(x)\,dx$ and $d\int_{\mathcal{O}} Z_t^\epsilon(x)p_t(x)\,dx$, while the formal proof goes through Yosida approximations of A.

Proposition 4.2 *We have*

$$
J(u^\epsilon) - J(u) = \mathbb{E}\int_0^T\int_{\mathcal{O}} \big[\delta l_t(x) + \delta b_t(x)p_t(x) + \delta\sigma_{jt}(x)q_{jt}(x)\big]\,dx\,dt + \frac12\Delta_3^\epsilon + o(\epsilon), \tag{4.1}
$$

where

$$
\Delta_3^\epsilon = \mathbb{E}\int_0^T\int_{\mathcal{O}} \bar H_t(x)\,Y_t^\epsilon(x)^2 dxdt + \mathbb{E}\int_{\mathcal{O}} \bar h(x)\,Y_T^\epsilon(x)^2\,dx, \tag{4.2}
$$

with

$$
\bar H_t(x) = l''(x,\bar X_t(x),\bar u_t) + b''(x,\bar X_t(x),\bar u_t)p_t(x) + \sigma_j''(x,\bar X_t(x),\bar u_t)q_{jt}(x), \quad \bar h(x) = h''(x,\bar X_T(x)).
$$

We notice that the multiplication by $\bar H_t(\cdot)$ is not a bounded operator in H.

Definition 4.1 *For fixed $t \in [0,T]$ and $f \in L^4(\mathcal{O})$, we consider the equation (understood as usual in mild form)*

$$\begin{cases} dY_s^{t,f}(x) = AY_s^{t,f}(x)\,ds + b'(x,\bar{X}_s(x),\bar{u}_s)Y_s^{t,f}(x)\,ds + \sigma_j'(x,\bar{X}_s(x),\bar{u}_s)Y_s^{t,f}(x)\,dW_s^j, & s \in [t,T], \\ Y_t^{t,f}(x) = f(x). \end{cases} \tag{4.3}$$

We denote $\mathcal{L}$ the space of bounded linear operators $L^4(\mathcal{O}) \to L^4(\mathcal{O})^* = L^{4/3}(\mathcal{O})$ and define a progressive process $(P_t)_{t\in[0,T]}$ with values in $\mathcal{L}$ setting for $t \in [0,T]$, $f,g \in L^4(\mathcal{O})$,

$$\langle P_t f, g\rangle = \mathbb{E}^{\mathcal{F}_t}\int_t^T\int_{\mathcal{O}} \bar{H}_s(x)Y_s^{t,f}(x)Y_s^{t,g}(x)\,dx\,ds + \mathbb{E}^{\mathcal{F}_t}\int_{\mathcal{O}} \bar{h}(x)Y_T^{t,f}(x)Y_T^{t,g}(x)\,dx \qquad \mathbb{P}-a.s.$$

(by abuse of language by $\langle\cdot,\cdot\rangle$ we also denote the duality between $L^4(\mathcal{O})$ and $L^{4/3}(\mathcal{O})$).
Exploiting the analyticity of the semigroup generated by A on $L^4(\mathcal{O})$ we prove the following proposition that is the key point for our final argument.

Proposition 4.3 *We have $\sup_{t\in[0,T]}\mathbb{E}\|P_t\|_{\mathcal{L}}^2 < \infty$. Moreover $\mathbb{E}|\langle (P_{t+\epsilon}-P_t)f,g\rangle| \to 0$, as $\epsilon \to 0$, $\forall f,g \in L^4(\mathcal{O})$. Finally for every $\eta \in (0,1/4)$ there exists a constant C_η such that*

$$|\langle P_t(-A)^\eta f, (-A)^\eta g\rangle| \le C_\eta \|f\|_4\|g\|_4(T-t)^{-2\eta}\left[\left(\int_t^T \mathbb{E}^{\mathcal{F}_t}|\bar{H}_s|^2 ds\right)^{1/2} + \left(\mathbb{E}^{\mathcal{F}_t}|\bar{h}|^2\right)^{1/2}\right],\ \mathbb{P}-a.s. \tag{4.4}$$

where $D(-A)^\eta$ is the domain of the fractional power of A in $L^4(\mathcal{O})$ and by $\|\cdot\|_4$ we denote the norm in $L^4(\mathcal{O})$.

5 The Maximum Principle

For $u \in U$ and $X, p, q_1, \ldots, q_m \in L^2(\mathcal{O})$ denote

$$\mathcal{H}(u,X,p,q_1,\ldots,q_m) = \int_{\mathcal{O}}\Big[l(x,X(x),u) + b(x,X(x),u)p(x) + \sigma_j(x,X(x),u)q_j(x)\Big]\,dx$$

Theorem 5.1 *Let $(\bar{X}_t,\bar{u}_t)$ be an optimal pair and let $p,q_1,\ldots,q_m$ be defined as in Proposition 4.1 and P be defined as in Definition 4.1. Then the following inequality holds $\mathbb{P}$-a.s. for a.e. $t \in [0,T]$ and for every $v \in U$:*

$$\begin{aligned}&\mathcal{H}(v,\bar{X}_t,p_t,q_{1t},\ldots,q_{mt}) - \mathcal{H}(\bar{u}_t,\bar{X}_t,p_t,q_{1t},\ldots,q_{mt})\\ &\quad+\frac{1}{2}\langle P_t[\sigma_j(\cdot,\bar{X}_t(\cdot),v)-\sigma_j(\cdot,\bar{X}_t(\cdot),\bar{u}_t)], \sigma_j(\cdot,\bar{X}_t(\cdot),v)-\sigma_j(\cdot,\bar{X}_t(\cdot),\bar{u}_t)\rangle \ge 0.\end{aligned}$$

Proof. By the Markov property of the solutions to equation (4.3) and Proposition 3.1 we get:

$$\begin{aligned}\mathbb{E}\int_0^T\langle\bar{H}_sY_s^\epsilon,Y_s^\epsilon\rangle\,ds + \mathbb{E}\langle\bar{h}Y_T^\epsilon,Y_T^\epsilon\rangle &= \mathbb{E}\int_{t_0+\epsilon}^T\langle\bar{H}_sY_s^{t_0+\epsilon,Y_{t_0+\epsilon}^\epsilon},Y_s^{t_0+\epsilon,Y_{t_0+\epsilon}^\epsilon}\rangle\,ds + \mathbb{E}\langle\bar{h}Y_T^{t_0+\epsilon,Y_{t_0+\epsilon}^\epsilon},Y_T^{t_0+\epsilon,Y_{t_0+\epsilon}^\epsilon}\rangle + o(\epsilon)\\ &= \mathbb{E}\langle P_{t_0+\epsilon}Y_{t_0+\epsilon}^\epsilon, Y_{t_0+\epsilon}^\epsilon\rangle + o(\epsilon).\end{aligned} \tag{5.1}$$

We wish to replace $P_{t_0+\epsilon}$ by P_{t_0} in the above that is we claim that $\mathbb{E}\langle(P_{t_0+\epsilon}-P_{t_0})\,Y_{t_0+\epsilon}^\epsilon,Y_{t_0+\epsilon}^\epsilon\rangle = o(\epsilon)$, or equivalently that $\mathbb{E}\langle(P_{t_0+\epsilon}-P_{t_0})\,\epsilon^{-1/2}Y_{t_0+\epsilon}^\epsilon,\epsilon^{-1/2}Y_{t_0+\epsilon}^\epsilon\rangle \to 0$.

To prove the above claim we need a compactness argument. A similar argument will allow us to approximate P by suitable finite dimensional projections at the end of this proof.
By the Markov inequality and Proposition 3.1 if we set $K_\delta = \{f \in L^4 \ :\ f \in D(-A)^\eta, \|f\|_{D(-A)^\eta} \le C_0\delta^{-1/4}\}$, for a suitable constant C_0, and denote by $\Omega_{\delta,\epsilon}$ the event $\{\epsilon^{-1/2}(-A)^{-\eta}Y_{t_0+\epsilon}^\epsilon \in K_\delta\}$ we get $\mathbb{P}(\Omega_{\delta,\epsilon}^c) \le \delta$.
Moreover

$$\begin{aligned}&\mathbb{E}\langle(P_{t_0+\epsilon}-P_{t_0})\,\epsilon^{-1/2}Y_{t_0+\epsilon}^\epsilon,\epsilon^{-1/2}Y_{t_0+\epsilon}^\epsilon\rangle\\ &= \mathbb{E}[\langle(P_{t_0+\epsilon}-P_{t_0})\,\epsilon^{-1/2}Y_{t_0+\epsilon}^\epsilon,\epsilon^{-1/2}Y_{t_0+\epsilon}^\epsilon\rangle 1_{\Omega_{\delta,\epsilon}^c}] + \mathbb{E}[\langle(P_{t_0+\epsilon}-P_{t_0})\,\epsilon^{-1/2}Y_{t_0+\epsilon}^\epsilon,\epsilon^{-1/2}Y_{t_0+\epsilon}^\epsilon\rangle 1_{\Omega_{\delta,\epsilon}}]\\ &=: A_1^\epsilon + A_2^\epsilon.\end{aligned}$$

By the Hölder inequality

$$|A_1^\epsilon| \le (\mathbb{E}\|P_{t_0+\epsilon}-P_{t_0}\|_{\mathcal{L}}^2)^{1/2}(\mathbb{E}\|\epsilon^{-1/2}Y_{t_0+\epsilon}^\epsilon\|_4^8)^{1/4}\mathbb{P}(\Omega_{\delta,\epsilon}^c)^{1/4},$$

and from the estimates in Proposition 4.3 we conclude that $|A_1^\epsilon| \le c\mathbb{P}(\Omega_{\delta,\epsilon}^c)^{1/4} = O(\delta^{1/4})$
On the other hand, recalling the definition of $\Omega_{\delta,\epsilon}$, $|A_2^\epsilon| \le \mathbb{E}\sup_{f\in K_\delta} |\langle (P_{t_0+\epsilon} - P_{t_0})(-A)^\eta f, (-A)^\eta f\rangle 1_{\Omega_{\delta,\epsilon}}|$.
Since K_δ is compact in L^4, it can be covered by a finite number N_δ of open balls with radius δ and centers denoted f_i^δ, $i = 1, \dots, N_\delta$. Since $D(-A)^\eta$ is dense in L^4, we can assume that $f_i^\delta \in D(-A)^\eta$. Given $f \in K_\delta$, let i be such that $\|f - f_i^\delta\|_4 < \delta$; then writing

$$\begin{aligned}&\langle (P_{t_0+\epsilon} - P_{t_0})(-A)^\eta f, (-A)^\eta f\rangle = \langle (P_{t_0+\epsilon} - P_{t_0})(-A)^\eta f_i^\delta, (-A)^\eta f_i^\delta\rangle\\ &-\langle (P_{t_0+\epsilon} - P_{t_0})(-A)^\eta (f - f_i^\delta), (-A)^\eta (f - f_i^\delta)\rangle + 2\langle (P_{t_0+\epsilon} - P_{t_0})(-A)^\eta f, (-A)^\eta (f - f_i^\delta)\rangle\end{aligned}$$

and taking expectation, it follows from (4.4) that

$$|A_2^\epsilon| \le \sum_{i=1}^{N_\delta} \mathbb{E}|\langle (P_{t_0+\epsilon} - P_{t_0})(-A)^\eta f_i^\delta, (-A)^\eta f_i^\delta\rangle| + c(T - t_0 - \epsilon)^{-2\eta}[\delta^2 + \delta^{3/4}],$$

and by the second statement in Proposition 4.3 we conclude that

$$\limsup_{\epsilon\downarrow 0} |A_2^\epsilon| \le c(T - t_0)^{-2\eta}[\delta^2 + \delta^{3/4}].$$

Letting $\delta \to 0$ we obtain $|A_1^\epsilon| + |A_2^\epsilon| \to 0$ and the proof that $\mathbb{E}\langle (P_{t_0+\epsilon} - P_{t_0})\,\epsilon^{-1/2} Y_{t_0+\epsilon}^\epsilon, \epsilon^{-1/2} Y_{t_0+\epsilon}^\epsilon\rangle \to 0$ is finished.

We come now to show that

$$\mathbb{E}\langle P_{t_0} Y_{t_0+\epsilon}^\epsilon, Y_{t_0+\epsilon}^\epsilon\rangle = \mathbb{E}\int_{t_0}^{t_0+\epsilon} \langle P_s \delta^\epsilon \sigma_j(s,\cdot), \delta^\epsilon \sigma_j(s,\cdot)\rangle\, ds + o(\epsilon).$$

If we treat A and P_{t_0} as bounded operators in H we get, by Itô rule:

$$\mathbb{E}\langle P_{t_0} Y_{t_0+\epsilon}^\epsilon, Y_{t_0+\epsilon}^\epsilon\rangle = 2\mathbb{E}\int_{t_0}^{t_0+\epsilon} \langle P_{t_0} Y_s^\epsilon, (A + b'(\bar X_t, \bar u_t)) Y_s^\epsilon\rangle ds + \mathbb{E}\int_{t_0}^{t_0+\epsilon} \langle P_{t_0} \delta^\epsilon \sigma_j(s,\cdot), \delta^\epsilon \sigma_j(s,\cdot)\rangle\, ds$$

and the claim follows recalling that $\mathbb{E}|Y_t^\epsilon|^2 = O(\epsilon)$ and the "continuity" of P stated in Proposition 4.3.
The general case is more technical and requires a double approximation: A by its Yosida Approximations and P by finite dimensional projections $P_t^N(\omega)f := \sum_{i,j=1}^N \langle P_t(\omega)e_i, e_j\rangle\langle e_i, f\rangle_2 e_j$, $f \in L^4$, where $(e_i)_{i\ge 1}$ is an orthonormal basis in L^2 which is also a Schauder basis of L^4.
The conclusion of the proof of the maximum principle is now standard (see, e.g. [2], [5] or [4]). We just have to write $J(u^\epsilon) - J(u)$ using (4.1), (4.2) and (5.1), to recall that $0 \le \epsilon^{-1}(J(u^\epsilon) - J(u))$ and to let $\epsilon \to 0$.

References

[1] A. Bensoussan, Stochastic maximum principle for distributed parameter systems. *J. Franklin Inst.* **315** (1983), no. 5-6, 387-406.

[2] Y. Hu and S. Peng, Adapted solution of a backward semilinear stochastic evolution equation. *Stochastic Anal. Appl.* **9** (1991), no. 4, 445-459.

[3] Q. Lu, and X. Zhang. General Pontryagin-Type Stochastic Maximum Principle and Backward Stochastic Evolution Equations in Infinite Dimensions. Preprint, (2012), arXiv:1204.3275v1.

[4] S. Peng, A general stochastic maximum principle for optimal control problems. *SIAM J. Control Optim.* **28**(1990), no. 4, pp. 966-979.

[5] S. J. Tang and X. J. Li, Maximum principle for optimal control of distributed parameter stochastic systems with random jumps. In: *Differential Equations, Dynamical Systems, and Control Science.* Lecture Notes in Pure and Appl. Math., **152** (1994), 867-890.